

# COUNTING SCHRÖDINGER BOUNDSTATES: SEMICLASSICS AND BEYOND

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**ABSTRACT.** This is a survey of the basic results on the behavior of the number of the eigenvalues of a Schrödinger operator, lying below its essential spectrum. We discuss both fast decaying potentials, for which this behavior is semiclassical, and slowly decaying potentials, for which the semiclassical rules are violated.

The outstanding personality of Sergey Lvovich Sobolev determined the development of Analysis in XX century in many aspects. One of his most influential contributions to Mathematics is the invention of the function spaces now named after him and the creation of the machinery of embedding theorems for these spaces. The ideology and the techniques based upon these theorems enabled S.L. Sobolev and his followers to find comprehensive and exact solutions to many key problems in Mathematical Physics. The paper to follow is devoted to a survey of results in one of such problems. This problem concerns the behavior of the discrete part of the spectrum of a Schrödinger operator with negative potential.

## 1. INTRODUCTION

The classical Weyl lemma states that the essential spectrum of a self-adjoint operator  $\mathbf{H}$  in a Hilbert space is stable under perturbations by a compact operator. This lemma has many important generalizations. In particular, if  $\mathbf{H}$  is non-negative, the result survives if the perturbation is only relatively compact with respect to  $\mathbf{H}$ , in the sense of quadratic forms.

The leading and most inspiring example in spectral theory, where the Weyl lemma plays the key role, concerns the discrete spectrum of a Schrödinger operator

$$\mathbf{H}_V = -\Delta - V$$

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on  $\mathbb{R}^d$ . Here  $V = V(x)$  is a real-valued measurable function on  $\mathbb{R}^d$  (the potential), which we assume to decay at infinity, in a certain appropriate sense. Then the operator can be defined via the corresponding quadratic form, considered on the Sobolev space  $H^1(\mathbb{R}^d)$ . We assume for simplicity that  $V \geq 0$ . Results for general real-valued potentials can be then derived by using the variational principle. In this paper we do not touch upon the results which take into account the interplay between the positive and the negative parts of the potential.

For the description of the spectrum of the operators involved we will use the following notation. Let  $\sigma(\mathbf{H})$  and  $E_{\mathbf{H}}(\cdot)$  stand for the spectrum and the spectral measure of a self-adjoint operator  $\mathbf{H}$ . We call the number

$$\text{bott}(\mathbf{H}) := \inf\{l : l \in \sigma(\mathbf{H})\}$$

the *bottom* of the operator  $\mathbf{H}$ . We put

$$N_-(\beta; \mathbf{H}) = \dim E_{\mathbf{H}}(-\infty, \beta), \quad \beta \in \mathbb{R}.$$

The relation

$$N_-(\beta; \mathbf{H}) < \infty$$

means that the spectrum of  $\mathbf{H}$  on the half-line  $(-\infty, \beta)$  is discrete, moreover, finite, and  $N_-(\beta; \mathbf{H})$  gives the number of the eigenvalues of  $\mathbf{H}$ , counted according to their multiplicities and lying on this half-line.

The spectrum of the free Laplacian  $\mathbf{H}_0 = -\Delta$  in  $L^2(\mathbb{R}^d)$  is the half-line  $[0, \infty)$ , and by the Weyl lemma the negative spectrum of  $\mathbf{H}_V$  is discrete. However, this lemma gives no quantitative information about the negative spectrum: it does not allow one to find out, whether the quantity  $N_-(0; \mathbf{H}_V)$  is infinite, or finite, and in the latter case it gives no control of its size. It is often important to answer these questions. In order to make the problem more transparent, it is useful to insert a real positive parameter (the *coupling constant*), and to study the above questions for the family

$$\mathbf{H}_{\alpha V} = -\Delta - \alpha V, \quad \alpha > 0. \quad (1.1)$$

The function  $N_-(0; \mathbf{H}_{\alpha V})$  grows together with  $\alpha$ , and this growth of the number of negative eigenvalues can be interpreted as birth of new bound states from the edge of the continuous spectrum as the exterior field grows. At the same time,  $N(0; \mathbf{H}_{\alpha V}) = N(0; -\alpha^{-1}\Delta - V)$ , so the behavior of this quantity as  $\alpha \rightarrow \infty$  describes simultaneously the semiclassical behavior of the eigenvalues as the 'Planck constant'  $\alpha^{-\frac{1}{2}}$  tends to 0.

Along with  $N_-(0; \mathbf{H}_{\alpha V})$ , one often studies the function  $N_-(-\gamma; \mathbf{H}_{\alpha V})$ , where  $\gamma > 0$ . If the assumptions about  $V$  guarantee discreteness of the

negative spectrum of  $\mathbf{H}_{\alpha V}$ , then the latter number is always finite. If, in addition,  $N_-(0; \mathbf{H}_{\alpha V}) = \infty$ , the behavior of  $N_-(-\gamma; \mathbf{H}_{\alpha V})$  as  $\gamma \rightarrow 0+$ , for  $\alpha$  fixed, is an important characteristics of the operator.

The main contents of the present paper is a survey and a certain detailing of the known results on the behavior of the function  $N_-(-\gamma; \mathbf{H}_{\alpha V})$ ,  $\gamma \geq 0$ , for the Schrödinger operator (1.1) and its generalizations – such as Schrödinger operators on manifolds, or in domains  $\Omega \subset \mathbb{R}^d$ . Note that in the latter cases the bottom of the Laplacian is not necessarily equal to zero. Then we discuss the behavior of  $N_-(\beta; \mathbf{H}_{\alpha V})$  for a fixed value of  $\beta \leq \text{bott}(-\Delta)$  (we refrain from using the notation  $N_-(-\gamma, \mathbf{H}_{\alpha V})$  except for the cases where  $\text{bott}(\mathbf{H}_0) = 0$ ).

Our starting point is the Weyl asymptotic law, which allows one to realize what sort of results is desirable.

If the potential  $V$  is nice (say,  $C_0^\infty$ ), then for any  $\gamma \geq 0$  the function  $N_-(-\gamma; \mathbf{H}_{\alpha V})$  exhibits the semiclassical, or Weyl, asymptotic behavior, that is,

$$N_-(-\gamma; \mathbf{H}_{\alpha V}) \sim w_d \alpha^{\frac{d}{2}} \int_{\mathbb{R}^d} V^{\frac{d}{2}} dx, \quad \alpha \rightarrow \infty. \quad (1.2)$$

Here  $w_d = v_d(2\pi)^{-d}$ , where  $v_d$  stands for the volume of the unit ball in  $\mathbb{R}^d$ . (The word 'semiclassical' is used in order to indicate that the expression on the right-hand side in (1.2) is proportional to the volume of the region in the classical phase space  $\mathbb{R}^{2d}$  where the classical Hamiltonian  $p^2 - \alpha V(x)$  is negative.) In particular, the asymptotic formula (1.2) hints that for any potential  $V \in L_{\text{loc}}^1(\mathbb{R}^d)$  the function  $N_-(-\gamma; \mathbf{H}_{\alpha V})$  cannot grow (in  $\alpha$ ) slower than  $O(\alpha^{\frac{d}{2}})$ . But can it grow faster?

In this connection, the following questions arise in a natural way.

**A. To describe the classes of potentials that guarantee the estimate**

$$N_-(-\gamma; \mathbf{H}_{\alpha V}) = O(\alpha^{\frac{d}{2}}), \quad \alpha \rightarrow \infty. \quad (1.3)$$

Another important question is this:

**B. Suppose that for a given potential  $V$  we have (1.3). Does this imply the asymptotic formula (1.2)?**

One more natural question:

**C. What can be said about the eigenvalues for such potentials that the negative spectrum of  $\mathbf{H}_{\alpha V}$  is still discrete, but (1.3) is violated?**

In the paper we discuss the present situation with answers to these questions. The answers heavily depend on the dimension. In particular, the answer to the question **B** is YES if  $d \geq 3$ , and it is NO if  $d = 1, 2$ .

We also discuss the analogues of these problems for the Laplacian on a manifold and, more briefly, on domains  $\Omega \subset \mathbb{R}^d$  and on the lattice  $\mathbb{Z}^d$ . Note that in all these cases the situation is understood up to a much lesser extent, than for  $\mathbb{R}^d$ .

The number  $N_-(0; \mathbf{H}_V)$  can be interpreted as the borderline value, for  $r = 0$ , of the quantity

$$S_r(V) = \sum_{l_j(\mathbf{H}_V) < 0} |l_j(\mathbf{H}_V)|^r, \quad r > 0.$$

Estimating such sums is important for Physics, and this is the main subject in the so called Lieb – Thirring inequalities. In this paper we do not touch upon this popular topic; see [15] for a survey and [11] for newer results.

## 2. OPERATORS ON $\mathbb{R}^d$ , $d \geq 3$

**2.1. The RLC estimate.** In the case considered, the answer to the questions **A**, **B** is given by the so called Rozenblum – Lieb – Cwikel estimate (the RLC estimate), named after the mathematicians who gave the first independent proofs of the result. In the form given below the result is due to Rozenblum [22, 23]. Other authors, see [18] and [6], did not discuss the necessity of the condition on  $V$ .

**Theorem 2.1.** *Let  $d \geq 3$ . Then there exists a constant  $C = C(d)$  such that for any  $V \in L^{\frac{d}{2}}(\mathbb{R}^d)$ ,  $V \geq 0$ , and any  $\gamma \geq 0$*

$$N_-(-\gamma; \mathbf{H}_{\alpha V}) \leq C(d) \alpha^{\frac{d}{2}} \int_{\mathbb{R}^d} V^{\frac{d}{2}} dx, \quad (2.1)$$

*and moreover, the asymptotic formula (1.2) holds.*

*Conversely, suppose that  $d \geq 3$ , for a certain  $V \geq 0$  the operator  $\mathbf{H}_{\alpha V}$  is well defined for all  $\alpha > 0$ , and for some  $\gamma \geq 0$  the function  $N_-(-\gamma; \mathbf{H}_{\alpha V})$  is  $O(\alpha^{\frac{d}{2}})$  as  $\alpha \rightarrow \infty$ . Then  $V \in L^{\frac{d}{2}}(\mathbb{R}^d)$ , and, therefore, estimate (2.1) and asymptotic formula (1.2) are fulfilled for an arbitrary  $\gamma \geq 0$ .*

Evidently, estimate (2.1) for any  $\gamma > 0$  and any  $\alpha > 0$  is a consequence of its particular case for  $\gamma = 0$  and  $\alpha = 1$ . Asymptotic formula (1.2) is proved first by elementary methods (Dirichlet – Neumann bracketing) for potentials  $V \in C_0^\infty(\mathbb{R}^d)$ . It extends to the general

case by a machinery, known as 'completion of spectral asymptotics' and presented in detail in the book [3], see especially Lemma 1.19 there.

The proofs given by Rozenblum, by Lieb, and by Cwikel, used different techniques. Rozenblum's approach was based upon the Sobolev embedding theorem in combination with Besicovitch-type covering theorem; Cwikel applied harmonic analysis and theory of interpolation of linear operators. Both these proofs apply to much more general classes of operators than just to the Laplacian, but only in the  $\mathbb{R}^d$ -setting. The first proof which admits generalization to other situations, say to operators on manifolds, is due to Lieb, who used the semigroup theory, in the form of path integrals.

Later several other proofs were suggested, including the ones given by Fefferman [10] and by Li and Yau [17]. For us, the latter is especially remarkable, since it shows in an extremely transparent form the deep connection between the 'global Sobolev inequality' and the RLC estimate. The techniques in [17] uses semigroup theory in a somewhat more direct way than in [18]. Like Lieb's proof, it admits far-reaching generalizations.

### 3. THE GENERAL RLC INEQUALITY

**3.1. The approach by Li - Yau.** What we present below, is an abstract version of the Li-Yau result. It was established in the paper [16], whose authors aimed at finding the most general setting in which the approach of [17] applies. The classical notion of sub-Markov semigroup is used in the formulation.

Let  $(\Omega, \sigma)$  be a measure space with sigma-finite measure. We denote  $L^q(\Omega) = L^q(\Omega, \sigma)$  and  $\|\cdot\|_q = \|\cdot\|_{L^q(\Omega)}$ . Suppose that a non-negative quadratic form  $Q[u]$  is defined on a dense in  $L^2(\Omega)$  linear subset  $\text{Dom}[Q]$ . We assume that  $Q$  is closed and that the corresponding self-adjoint operator  $\mathbf{A} = \mathbf{A}_Q$  generates a symmetric, positivity preserving semigroup. In this situation we say that the operator  $\mathbf{A}$  is a *sub-Markov generator*. We also suppose that there exist an exponent  $q > 2$  and a positive constant  $K$ , such that

$$\|u\|_q^2 \leq KQ[u], \quad \forall u \in \text{Dom } Q. \quad (3.1)$$

**Theorem 3.1.** *Let  $Q[u]$  be the quadratic form of a sub-Markov generator in  $L^2(\Omega)$ . Suppose that estimate (3.1) is satisfied, with some  $q > 2$ . Let*

$$0 \leq V \in L_p(\Omega), \quad p = \left(1 - \frac{2}{q}\right)^{-1}. \quad (3.2)$$

Then the quadratic form

$$Q_V[u] := Q[u] - \int_{\Omega} V|u|^2 d\sigma, \quad u \in \text{Dom}[Q],$$

is bounded from below in  $L^2(\Omega)$  and closed. The negative spectrum of the corresponding self-adjoint operator  $\mathbf{A} - V$  in  $L^2(\Omega)$  is finite, and

$$N_-(0; \mathbf{A} - V) \leq C(p)K^p \int_{\Omega} V^p d\sigma, \quad C(p) = e^2 \left(\frac{p}{2}\right)^p. \quad (3.3)$$

We will call (3.3) the *general RLC inequality*.

It is well known that for any  $d$  the (minus) Laplacian on  $\mathbb{R}^d$  is a sub-Markov generator. The inequality (3.1) is satisfied if  $d \geq 3$ , with  $q = \frac{2d}{d-2}$ , so that in (3.2) we have  $p = \frac{d}{2}$ . This is the so-called ‘global Sobolev inequality’, and the sharp value of the constant  $K$  is known, see, e.g., [19], inequality (3) in Section 2.3.3. So, Theorem 3.1 implies the RLC estimate (2.1), with an explicitly given constant. For the case  $d = 3$ , which is the most interesting for Physics, this constant is slightly greater than the best value  $C(3) = .116$  in (2.1), known up to now. It should be compared with the constant  $w_3 = .078$  in the asymptotic formula (1.2). This best value is given by Lieb’s approach which we discuss in the next subsection. It is worth mentioning here that the sharp value of the constant  $C(d)$  in (2.1), even for  $d = 3$ , is up to now unknown.

**3.2. The approach by Lieb.** Below we present the main result of the paper [24], where an abstract version of Lieb’s approach was elaborated.

Any non-negative self-adjoint operator  $\mathbf{A}$  in  $L^2(\Omega)$  generates a contractive semigroup  $e^{-t\mathbf{A}}$ . We suppose that this semigroup is  $(2, \infty)$ -bounded, which means that for any  $t > 0$  the operator  $e^{-t\mathbf{A}}$  is bounded as acting from  $L^2(\Omega)$  to  $L^\infty(\Omega)$ . We write

$$\mathbf{A} \in \mathcal{P}$$

if the semigroup  $e^{-t\mathbf{A}}$  is  $(2, \infty)$ -bounded and positivity preserving.

Let  $K(t; x, y)$  be the integral (Schwartz) kernel of  $e^{-t\mathbf{A}}$ . Then the function  $K(t; x, x)$  is well-defined on  $\mathbb{R}_+ \times \Omega$ , and it belongs to  $L^\infty(\Omega)$  for each  $t > 0$ . We put

$$M_{\mathbf{A}}(t) = \|K(t; \cdot)\|_{\infty}.$$

The main result is a *parametric estimate*, see (3.5) below: it involves an arbitrary function  $G(z)$  of a certain class, as a parameter. The class  $\mathcal{G}$  of admissible functions  $G$  is defined as follows.

The function  $G$  is continuous, convex, non-negative, grows at infinity no faster than a polynomial, and is such that  $z^{-1}G(z)$  is integrable at zero. With each  $G \in \mathcal{G}$  we associate another function,

$$g(l) = \int_{\mathbb{R}_+} z^{-1}G(z)e^{-\frac{z}{l}}dz, \quad l > 0. \quad (3.4)$$

**Theorem 3.2.** *Suppose  $\mathbf{A} \in \mathcal{P}$  is such that the function  $M_{\mathbf{A}}(t)$  is integrable at infinity and is  $o(t^{-a})$  at zero, with some  $a > 0$ . Fix a function  $G \in \mathcal{G}$ , and define  $g(l)$  as in (3.4). Then*

$$N_-(0; \mathbf{A} - V) \leq \frac{1}{g(1)} \int_{\mathbb{R}_+} \frac{dt}{t} \int_{\Omega} M_{\mathbf{A}}(t)G(tV(x))d\sigma, \quad (3.5)$$

whenever the integral on the right is finite.

Note that finiteness of the integral in (3.5) guarantees that the relative bound of  $V$  with respect to the quadratic form of the operator  $\mathbf{A}$  is smaller than 1, so that the operator  $\mathbf{A} - V$  is well-defined via its quadratic form.

If  $(\Omega, \sigma)$  is  $\mathbb{R}^d$  with the Lebesgue measure, and  $\mathbf{A} = -\Delta$ , then the semigroup  $e^{-t\mathbf{A}}$  is positivity preserving and  $(2, \infty)$ -bounded, and  $M_{-\Delta}(t) = (2\pi)^{-\frac{d}{2}}t^{-\frac{d}{2}}$ . Since  $M(t)$  is a pure power, the choice of  $G \in \mathcal{G}$  is indifferent, within the value of the constant factor in the estimate. Indeed, by a change of variables the estimate (3.5) reduces to the form

$$N_-(0; \mathbf{A} - V) \leq C(G) \int_{\mathbb{R}^d} V^{\frac{d}{2}}dx, \quad (3.6)$$

where

$$C(G) = \frac{1}{g(1)(2\pi)^{\frac{d}{2}}} \int_0^\infty z^{-(\frac{d}{2}+1)}G(z)dz.$$

The assumptions about  $G$  and the finiteness of  $C(G)$  dictate the restriction  $d \geq 3$ . The optimal choice of  $G \in \mathcal{G}$  was pointed out by Lieb [18].

The relation between Theorems 3.2 and 3.1 is based upon the deep connection between the Sobolev type inequality (3.1) and the estimate

$$M_{\mathbf{A}}(t) \leq Ct^{-\frac{d}{2}}; \quad t \in (0, \infty) \quad (3.7)$$

for the heat kernel corresponding to the operator  $\mathbf{A} = \mathbf{A}_Q$ . This connection was established by Varopoulos; see [26], Sect.II.2 or [7], Theorem 2.4.2.

**Theorem 3.3.** *If the quadratic form  $Q[u]$  generates a symmetric sub-Markov semigroup on the measure space  $(\Omega, \sigma)$ , then the estimate (3.7) with  $d > 2$  is equivalent to the inequality (3.1) with  $q = \frac{2d}{d-2}$ .*

So, the result of Theorem 3.2 yields the general RLC inequality (3.3) and thus, is stronger than Theorem 3.1. Indeed, in the general setting the behavior of the function  $M_{\mathbf{A}}(t)$  is not necessarily expressed by the inequality (3.7), with the same exponent  $d$  both as  $t \rightarrow 0$  and  $t \rightarrow \infty$ . In many cases, one has

$$M_{\mathbf{A}}(t) \leq C_0 t^{-\frac{\delta}{2}}, \quad t < 1; \quad (3.8)$$

$$M_{\mathbf{A}}(t) \leq C_{\infty} t^{-\frac{D}{2}}, \quad t > 1, \quad (3.9)$$

with  $D \neq \delta$ . In [26] such estimates were studied for the sub-Laplacian on nilpotent groups, and the numbers  $\delta, D$  were called there dimensions at zero, resp., at infinity. We will use these terms as well. One encounters a similar situation when studying the Laplacian on manifolds, or on domains in  $\mathbb{R}^d$ .

If the estimates (3.8), (3.9) are known with  $\delta, D > 2$ , the eigenvalue estimates obtained from (3.5) vary essentially, depending on which dimension,  $\delta$  or  $D$  is larger.

We formulate the corresponding results, not trying to find best possible constants, however we include the coupling parameter  $\alpha$ .

**Theorem 3.4.** *Under the conditions of Theorem 3.2 suppose that the inequalities (3.8), (3.9) are satisfied with some  $\delta, D > 2$ . Then the following eigenvalue estimates hold:*

$$N_{-}(0; \mathbf{H}_{\alpha V}) \leq C'_0 \alpha^{\frac{\delta}{2}} \int_{\Omega} V^{\frac{\delta}{2}} d\sigma + C'_{\infty} \alpha^{\frac{D}{2}} \int_{\Omega} V^{\frac{D}{2}} d\sigma, \quad (3.10)$$

if  $\delta \geq D$ , and

$$N_{-}(0; \mathbf{H}_{\alpha V}) \leq C''_0 \alpha^{\frac{\delta}{2}} \int_{\alpha V \geq 1} V^{\frac{\delta}{2}} d\sigma + C''_{\infty} \alpha^{\frac{D}{2}} \int_{\alpha V < 1} V^{\frac{D}{2}} d\sigma \quad (3.11)$$

if  $\delta \leq D$ .

*Remark 3.5.* If  $\delta \leq D$ , the inequalities (3.8) and (3.9) imply (3.7) with  $d = D$ , and hence

$$N_{-}(0; \mathbf{H}_{\alpha V}) \leq \tilde{C} \alpha^{\frac{D}{2}} \|V\|_{\frac{D}{2}}^{\frac{D}{2}}, \quad D > 2. \quad (3.12)$$

It is often important that the assumption  $\delta > 2$ , appearing in Theorem 3.4, here is unnecessary.

We discuss applications of the estimates (3.10) and (3.11) in Sections 7, 8, and 9.



#### 4. OPERATORS ON $\mathbb{R}^d$ , $d \geq 3$ : NON-SEMICLASSICAL BEHAVIOR OF $N_-(0; \mathbf{H}_{\alpha V})$ .

Suppose now that  $d \geq 3$  but  $V \notin L^{\frac{d}{2}}(\mathbb{R}^d)$ , though  $V(x)$  vanishes as  $|x| \rightarrow \infty$ , again in some appropriate sense. Then the negative spectrum of  $-\Delta - \alpha V$  is still discrete, but the RLC inequality becomes useless. In this situation some estimates for the quantity  $N_-(0; \mathbf{A} - \alpha V)$  can be obtained by using interpolation between the RLC inequality (2.1) and a remarkable result, due to Maz'ya [19], Section 2.3.3. This result gives the necessary and sufficient conditions on a weight function  $V \geq 0$  for the Hardy-type inequality

$$\int_{\mathbb{R}^d} V|u|^2 dx \leq C(V) \int_{\mathbb{R}^d} |\nabla u|^2 dx, \quad \forall u \in C_0^\infty(\mathbb{R}^d)$$

to be satisfied.

Here we present only a particular case of the general class of estimates obtained by this approach. See [4] for detail.

**Theorem 4.1.** *Let  $d \geq 3$ . Suppose that for some  $q > \frac{d}{2}$  the potential  $V$  satisfies the condition*

$$|V|_q^q := \sup_{t>0} \left( t^q \int_{|x|^2 V(x) > t} \frac{dx}{|x|^d} \right) < \infty. \quad (4.1)$$

*Then for any  $\alpha > 0$  the operator  $-\Delta - \alpha V$  on  $\mathbb{R}^d$  is bounded from below, its negative spectrum is finite, and the estimate*

$$N_-(0; \mathbf{H}_{\alpha V}) \leq C(d, q) \alpha^q |V|_q^q \quad (4.2)$$

*is satisfied.*

The condition (4.1) means that the function  $|x|^2 V(x)$  belongs to the so-called *weak  $L^q$ -space*, usually denoted by  $L_w^q$ , with respect to the measure  $|x|^{-d} dx$  on  $\mathbb{R}^d$ . The functional  $|V|_q$  is equivalent to the norm in this space, but it does not meet the triangle inequality itself. The space  $L_w^q$  is non-separable, and it contains the usual space  $L_q$  with respect to the same measure. Replacing in (4.2) the functional  $|V|_q$  by the norm in  $L^q$  coarsens the estimate, and we come to the inequality

$$N_-(0; \mathbf{H}_{\alpha V}) \leq C'(d, q) \alpha^q \int_{\mathbb{R}^d} V^q |x|^{2q-d} dx, \quad 2q > d, \quad (4.3)$$

which looks simpler than (4.2). The estimate (4.3) was established in [9] by a direct approach, generalizing the one in [23]. However, (4.3) is knowingly not exact: it is easy to see that the finiteness of the integral in (4.3) implies

$$N_-(0; \mathbf{H}_{\alpha V}) = o(\alpha^q), \quad \alpha \rightarrow \infty. \quad (4.4)$$

Indeed, this is certainly the case for the potentials  $V \in C_0^\infty(\mathbb{R}^d)$ . Such potentials are dense in  $L^q$  with weight  $|x|^{2q-d}$ , and the procedure of completion of spectral asymptotics, mentioned in the paragraph next to Theorem 2.1, shows that (4.4) extends to all  $V$  from this space. This nice reasoning is due to Birman (private communication). It easily extends to the general situation, and it shows that *any order-sharp estimate of order  $q > \frac{d}{2}$  for the quantity  $N_-(0; \mathbf{H}_{\alpha V})$  must involve some non-separable class of potentials*.

In contrast to (4.3), the estimate (4.2) is order-sharp: say, for the potential  $V$  which for large  $|x|$  is equal to

$$V(x) = |x|^{-2}(\log |x|)^{-\frac{1}{q}}, \quad 2q > d, \quad (4.5)$$

the condition (4.2) is satisfied, and for such potentials the asymptotics

$$N_-(0; \mathbf{H}_{\alpha V}) \sim c_q \alpha^q, \quad c_q > 0, \quad \alpha \rightarrow \infty$$

is known, see [4].

The condition (4.1) allows local singularities of  $V$  at the point  $x = 0$ , which are stronger than those allowed by the inclusion  $V \in L^{\frac{d}{2}}(\mathbb{R}^d)$ . The weight function  $|x|^2$  and the measure  $|x|^{-d}dx$  in (4.1) can be replaced by functions and measures in a rather wide class, see [4]. In particular, this allows one to control effects coming from singularities of  $V$  distributed on submanifolds in  $\mathbb{R}^d$ . For example, suppose we are interested in the potentials with singularities at the sphere  $|x| = 1$ . Then, instead of (4.2), one can use the estimate

$$N_-(0; \mathbf{H}_{\alpha V}) \leq C \alpha^q \sup_{t>0} \int_{V(x)|x|-1|^{\frac{2}{d}}>t} \frac{dx}{||x|-1|}, \quad 2q > d \geq 3.$$

Both this estimate and (4.2) are particular cases of Theorem 4.1 in [4].

We do not think that a unified condition on the potential, which is necessary and sufficient for  $N_-(0; \mathbf{H}_{\alpha V}) = O(\alpha^q)$  with a prescribed value of  $q > \frac{d}{2}$ , does exist.

## 5. OPERATORS ON THE SEMI-AXIS

**5.1. Semiclassical behavior.** In the case  $d = 1$  it is natural to deal with the operators on the semi-axis  $\mathbb{R}_+$ , defined as

$$\mathbf{H}_{\alpha V} u(x) = -u''(x) - \alpha V(x)u(x), \quad u(0) = 0. \quad (5.1)$$

An accurate definition can be given via the corresponding quadratic form. The case of operators on the whole axis is easily reduced to this one, by imposing the additional Dirichlet condition at  $x = 0$  and

adding up the two similar estimates for the operators acting on the positive and the negative semi-axis. The term  $+1$  must be included in the right-hand side of the resulting estimate, since imposing this boundary condition means the passage to a subspace of codimension 1 in  $H^1(\mathbb{R})$ . Appearing of the term  $+1$  reflects the fact that  $l = 0$  is a resonance point for the operator  $-\frac{d^2}{dx^2}$  in  $L^2(\mathbb{R})$ . This means that for an arbitrary non-trivial potential  $V \geq 0$  at least one eigenvalue exists for any  $\alpha > 0$ . Hence, no estimate homogeneous in  $\alpha$  is possible.

The character of estimates for the operator (5.1) is quite different from the RLC inequality which governs the case  $d \geq 3$ . The necessary and sufficient condition for the semiclassical order

$$N_-(0; \mathbf{H}_{\alpha V}) = O(\alpha^{\frac{1}{2}}) \quad (5.2)$$

is given by Theorem 5.1 below. However, this condition hardly can be re-formulated in purely function-theoretic terms.

With any function  $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}_+)$  we associate the sequence  $\boldsymbol{\eta}(V) = \{\eta_j(V)\}$ ,  $j \in \mathbb{Z}$ , where

$$\eta_j(V) = 2^j \int_{I_j} V(x) dx, \quad I_j = (2^j, 2^{j+1}), \quad j \in \mathbb{Z}. \quad (5.3)$$

It is not difficult to show that

$$N_-(0; \mathbf{H}_{\alpha V}) \leq C \alpha^{\frac{1}{2}} \sum_{j \in \mathbb{Z}} \eta_j^{\frac{1}{2}}(V),$$

so that the condition

$$\boldsymbol{\eta}(V) \in \ell^{\frac{1}{2}} \quad (5.4)$$

is sufficient for the estimate (5.2). It also guarantees validity of the Weyl asymptotics, which in this case takes the form

$$N_-(0; \mathbf{H}_{\alpha V}) \sim \pi^{-1} \alpha^{\frac{1}{2}} \int_{\mathbb{R}_+} V^{\frac{1}{2}} dx, \quad \alpha \rightarrow \infty. \quad (5.5)$$

However, the condition (5.4) is not necessary either for (5.2), or for (5.5).

In order to write the necessary and sufficient condition, let us consider the family of eigenvalue problems on the intervals  $I_j$ ,  $j \in \mathbb{Z}$ :

$$-lu''(x) = V(x)u(x) \text{ on } I_j, \quad u(2^j) = u(2^{j+1}) = 0. \quad (5.6)$$

Here it is convenient for us to put the spectral parameter in the left-hand side, then for each  $j$  the eigenvalues  $l_{j,k}$ ,  $k = 1, 2, \dots$ , of the problem (5.6) correspond to a compact operator. Let  $n_j(l)$  stand for their counting function:

$$n_j(l) = \#\{k : l_{j,k} > l\}, \quad l > 0.$$

Each function  $n_j(l)$  satisfies the estimate

$$l^{\frac{1}{2}} n_j(l) \leq C (2^j \eta_j(V))^{\frac{1}{2}} = C 2^j \left( \int_{I_j} V dx \right)^{\frac{1}{2}} \quad (5.7)$$

and exhibits the Weyl asymptotic behavior:

$$l^{\frac{1}{2}} n_j(l) \rightarrow \pi^{-1} \int_{I_j} V^{\frac{1}{2}} dx. \quad (5.8)$$

The estimate (5.7) is uniform in  $j$  (i.e., the constant  $C$  does not depend on  $j$ ), but the asymptotics (5.8) is not. This is reflected in the fact that the potential  $V$  is involved in (5.7) and in (5.8) in two different ways.

The following result was obtained in [20].

**Theorem 5.1.** *Let  $0 \leq V \in L_{loc}^1(\mathbb{R}_+)$ , and let  $\mathbf{H}_{\alpha V}$ ,  $\alpha > 0$ , be the family of operators (5.1). The two conditions*

$$\#\{j \in \mathbb{Z} : \eta_j(V) > l\} = O(l^{-\frac{1}{2}}), \quad l + l^{-1} \rightarrow \infty, \quad (5.9)$$

and

$$\sup_{l>0} \sum_j l^{\frac{1}{2}} n_j(l) < \infty \quad (5.10)$$

are necessary and sufficient for the semiclassical order (5.2) of the quantity  $N_-(0; \mathbf{H}_{\alpha V})$ .

The condition (5.9) means by definition that the sequence  $\boldsymbol{\eta}(V)$  belongs to the *weak  $\ell^{\frac{1}{2}}$ -space* (notation  $\ell_w^{\frac{1}{2}}$ ). This condition is much weaker than (5.4).

The conditions (5.9) and (5.10) do not guarantee the Weyl asymptotics (5.5). The necessary and sufficient condition on  $V$  for validity of this asymptotics was also established in [20]; we do not duplicate it here. Note only that in [20] a series of examples was constructed of potentials  $V$  for which the estimate (5.2) holds but the asymptotic formula is valid with the coefficient different from the one in (5.5). This is impossible in dimension  $d \geq 3$ .

**5.2. Non-semiclassical behavior of  $N_-(0; \mathbf{H}_{\alpha V})$ .** The situation here turns out to be rather simple. The criterium for  $N_-(0; \mathbf{H}_{\alpha V}) = O(\alpha^q)$  with a given  $q > \frac{1}{2}$  can be expressed in terms of the same sequence (5.3).

**Theorem 5.2.** *Let  $0 \leq V \in L_{loc}^1(\mathbb{R}_+)$ , and let  $2q > 1$ . The condition*

$$\#\{j \in \mathbb{Z} : \eta_j(V) > l\} = O(l^{-q}), \quad l + l^{-1} \rightarrow \infty, \quad (5.11)$$

is necessary and sufficient for  $N_-(0; \mathbf{H}_{\alpha V}) = O(\alpha^q)$ , and the inequality

$$N_-(0; \mathbf{H}_{\alpha V}) \leq C_q \alpha^q \sup_{l>0} l^q \#\{j \in \mathbb{Z} : \eta_j(V) > l\} \quad (5.12)$$

is satisfied.

The condition similar to (5.11), with  $o(l^{-q})$  on the right, is necessary and sufficient for  $N_-(0; \mathbf{H}_{\alpha V}) = o(\alpha^q)$ .

In particular, the condition (5.11) with  $q = 1$  is fulfilled, provided that

$$\int_{\mathbb{R}_+} xV(x)dx < \infty.$$

The inequality

$$N_-(0; \mathbf{H}_{\alpha V}) \leq \alpha \int_{\mathbb{R}_+} xV(x)dx$$

is the classical Bargmann estimate, see, e.g., [21]. So, the inequality (5.12) covers this result, within the value of the constant factor. Note that under the Bargmann condition one always has  $N_-(0; \mathbf{H}_{\alpha V}) = o(\alpha)$ . The argument is the same as in Section 4.

The proof of Theorem 5.2 can be found in [3], where actually more general multi-dimensional problems were analyzed, and in [2]. See also [1], where the result is presented without proof.

Theorem 5.2 turns out to be quite useful for the estimation of  $N_-(\gamma; \mathbf{H}_{\alpha V})$  for such multi-dimensional problems where an additional ‘channel’ can be singled out, that contributes independently to the behavior of this function for large values of  $\alpha$ . This happens, for instance, in many problems on manifolds, see Section 8. Another, may be the most striking example, is connected with the Laplacian on  $\mathbb{R}^2$ . We discuss this case in the next section.

## 6. OPERATORS ON $\mathbb{R}^2$

**6.1. Semiclassical behavior.** In the borderline case  $d = 2$  the exhaustive description of the class of potentials such that  $N_-(0; \mathbf{H}_{\alpha V}) = O(\alpha)$ , or at least

$$N_-(-\gamma; \mathbf{H}_{\alpha V}) = O(\alpha), \quad \forall \gamma > 0, \quad (6.1)$$

is not known up to present. On the technical level, this is a consequence of the fact that the embedding theorem  $H^1(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ ,  $q = \frac{2d}{d-2}$ , fails for  $d = 2$  (when  $q = \infty$ ), or of the equivalent fact that  $M_\Delta(t) = ct^{-1}$ , see Subsection 3.2, so the integral in (3.5) diverges. There are various sufficient conditions on the potential which ensure the order-sharp in  $\alpha$  estimate for the function (6.1), but all of them are not sharp

in the function classes for  $V$ . Even the most general sufficient condition of this type, known up to now (formulated in terms of Orlicz spaces), see [25], is not necessary. What is more, there are problems of a rather close nature, for which the RLC-like condition  $V \in L^1(\mathbb{R}^2)$  turns out to be sufficient, see [13, 14]. So, for  $d = 2$  the situation is not well understood up to now.

Below we present a comparatively simple sufficient condition, which was found in [1]. Fix a number  $\varkappa > 1$ , and with any potential  $0 \leq V \in L^{\varkappa}_{\text{loc}}(\mathbb{R}^2)$  let us associate the sequence  $\boldsymbol{\theta}(V, \varkappa) = \{\theta_j(V, \varkappa)\}$ ,  $j = 0, 1, \dots$ , where

$$\begin{aligned}\theta_0(V, \varkappa) &= \int_{|x| < 1} V^{\varkappa} dx, \\ \theta_j(V, \varkappa) &= \int_{2^{j-1} < |x| < 2^j} |x|^{2(\varkappa-1)} V^{\varkappa} dx, \quad j \in \mathbb{N}.\end{aligned}$$

**Theorem 6.1.** *For any fixed numbers  $\varkappa > 1$  and  $\gamma > 0$  there exists a constant  $C(\gamma, \varkappa) > 0$  such that as soon as*

$$\boldsymbol{\theta}(V, \varkappa) \in \ell^1, \quad (6.2)$$

*the operator  $\mathbf{H}_{\alpha V}$  is bounded from below for any  $\alpha > 0$ , its negative spectrum is discrete, and*

$$N_-(-\gamma; \mathbf{H}_{\alpha V}) \leq C(\gamma, \varkappa) \alpha \|\boldsymbol{\theta}(V, \varkappa)\|_1.$$

The constant  $C(\gamma, \varkappa)$  may blow up as  $\gamma \rightarrow 0$ , and the assumption (6.2) does not guarantee the semiclassical behavior of  $N_-(0; \mathbf{H}_{\alpha V})$ . It turns out that for the analysis of this behavior one has to consider separately the subspace  $\mathcal{F}$  of radial functions,  $u(x) = f(|x|)$ . On  $\mathcal{F}$  the quadratic form of  $\mathbf{H}_{\alpha V}$  generates a second order ordinary differential operator whose spectrum is not controlled by the sequence (4.1). In order to control it and to have the semiclassical order  $N_-(0; \mathbf{H}_{\alpha V}) = O(\alpha)$  for the original operator, one uses Theorem 5.2 with the exponent  $q = 1$ . In the next theorem we present the final result which can be obtained by means of this approach; in formula (6.3) below we express the potential  $V$  in the polar coordinates.

**Theorem 6.2.** *Let  $V \geq 0$  satisfy the conditions of Theorem 6.1. Consider an auxiliary ‘effective potential’ on  $\mathbb{R}_+$ ,*

$$F_V(t) = e^{2t} \int_{-\pi}^{\pi} V(e^t, \phi) d\phi, \quad t > 0. \quad (6.3)$$

*Let  $\{\eta_j(V)\}$  denote the sequence (5.3) for the potential  $F_V$ . Then (6.1) holds also for  $\gamma = 0$  if and only if the additional condition (5.11) with  $q = 1$  is fulfilled.*

Note that by changing (5.11) to a stronger condition, with  $o(l^{-1})$  on the right, we come to a condition ensuring the Weyl asymptotics (1.2) for  $d = 2$ . See the papers [25] and, especially, [1] for more detail and for discussion.

This effect (appearance of an additional differential operator in a lower dimension, which contributes to the behavior of  $N_-(0; \mathbf{H}_{\alpha V})$  in an independent way) we meet in several other problems, discussed in Section 8. This can be interpreted as opening of an additional channel which affects the behavior of the system studied.

**6.2. Non-semiclassical behavior.** It is easy to see that the condition  $\theta(V, \varkappa) \in \ell^\infty$  is sufficient for form-boundedness in  $H^1(\mathbb{R}^2)$  of the multiplication by  $V$ . The next result follows from this property and from Theorems 6.1, 6.2 by interpolation.

**Theorem 6.3.** *1. Suppose that for some  $q > 1$  we have*

$$\#\{j \in \mathbb{N} : \theta_j(V, \varkappa) > l\} = O(l^{-q}).$$

*Then for any  $\gamma > 0$  and  $\alpha > 0$*

$$N_-(-\gamma; \mathbf{H}_{\alpha V}) \leq 1 + C_{\gamma, q} \alpha^q \sup_{l > 0} l^q \#\{j \in \mathbb{N} : \theta_j(V, \varkappa) > l\}. \quad (6.4)$$

*2. Besides, suppose that the sequence  $\eta_j(V)$ , introduced in Theorem 6.2, satisfies the condition (5.11), with the same value of  $q$ . Then  $N_-(0; \mathbf{H}_{\alpha V}) = O(\alpha^q)$ , and the function  $N_-(0; \mathbf{H}_{\alpha V})$  is controlled by the expression as in the left-hand side of (6.4), plus the additional term*

$$\alpha^q \sup_{l > 0} l^q \#\{j \in \mathbb{Z} : \eta_j(V, \varkappa) > l\}.$$

## 7. SCHRÖDINGER OPERATOR ON MANIFOLDS

**7.1. Preliminary remarks.** Let  $\mathcal{M} = \mathcal{M}^d$  be a smooth Riemannian manifold of dimension  $d \geq 3$ , and let  $dx$  stand for the volume element on  $\mathcal{M}$ . In this section we discuss the behavior of the function  $N_-(\beta; -\Delta_{\mathcal{M}} - \alpha V)$  where  $\Delta_{\mathcal{M}}$  is the Laplacian on  $\mathcal{M}$  (i.e., the Laplace-Beltrami operator). In order to avoid any ambiguity, here we do not use the shortened notation  $\mathbf{H}_{\alpha V}$ . As a rule, we suppose that  $\mathcal{M}$  is non-compact. Otherwise, the spectrum of  $-\Delta_{\mathcal{M}}$  is discrete, and it makes no sense to speak about birth of eigenvalues of  $-\Delta_{\mathcal{M}} - \alpha V$  from the essential spectrum of  $-\Delta_{\mathcal{M}}$ .

For a complete Riemannian manifold  $\mathcal{M}$  the operator  $-\Delta_{\mathcal{M}}$ , defined initially on  $C_0^\infty(\mathcal{M})$ , is essentially self-adjoint and generates a sub-Markov semigroup. Thus, the results of Theorems 3.1 and 3.2 can be applied as soon as one has sufficient information about the embedding theorem on  $\mathcal{M}$ , or about estimates of the heat kernel. The global

Sobolev inequality (3.1) with the correct order  $q = 2d(d-2)^{-1}$  holds only in some special cases, and for general manifolds, probably, the only existing approach is based upon heat kernel estimates of the type (3.8), (3.9). Usually (though, not always) (3.8) is satisfied with  $\delta = d$ . For example, this is the case for the manifolds of bounded geometry, see, e.g., [12]. In the discussion below we will assume that

$$M_{-\Delta_{\mathcal{M}}}(t) \leq C_0 t^{-\frac{d}{2}}, \quad t < 1. \quad (7.1)$$

On the other hand,  $D$  in (3.9) reflects the global geometry of the manifolds, however, rather roughly, and any relation  $d > D$ ,  $d = D$ , or  $d < D$  is possible. The results that follow from such estimates are given by Theorem 3.4, where one should take  $\delta = d$ .

An important difference from the case  $\mathcal{M} = \mathbb{R}^d$  is that now the possibility of a positive  $\beta_{\mathcal{M}} := \text{bott}(-\Delta_{\mathcal{M}})$  is not excluded. May be, the only general result which holds true for any manifold subject to (7.1) is the following elementary, but useful statement.

**Theorem 7.1.** *Let  $\mathcal{M} = \mathcal{M}^d$  be a smooth complete Riemannian manifold,  $d \geq 3$ . Suppose that the inequality (7.1) is satisfied. Then for any  $0 \leq V \in L^{\frac{d}{2}}(\mathcal{M})$  and for any  $\beta < \beta_{\mathcal{M}}$  the following inequality holds:*

$$N_-(\beta; -\Delta_{\mathcal{M}} - \alpha V) \leq C(\mathcal{M}, \beta) \alpha^{\frac{d}{2}} \int_{\mathcal{M}} V^{\frac{d}{2}} dx, \quad \forall \alpha > 0. \quad (7.2)$$

Along with the estimate (7.2), the Weyl asymptotic formula

$$N_-(\beta; -\Delta_{\mathcal{M}} - \alpha V) \sim w_d \alpha^{\frac{d}{2}} \int_{\mathcal{M}} V^{\frac{d}{2}} dx, \quad \alpha \rightarrow \infty$$

is satisfied.

We only outline the proof of inequality (7.2). For any  $\beta < 0$ , the semigroup  $e^{-t(-\Delta_{\mathcal{M}} - \beta)}$  is sub-Markov (together with  $e^{t\Delta_{\mathcal{M}}}$ ), and the function  $M_{-\Delta_{\mathcal{M}} - \beta}(t) = e^{\beta t} M_{\Delta_{\mathcal{M}}}(t)$  satisfies the same estimate (7.2). Besides, this function exponentially decays as  $t \rightarrow \infty$ , and hence (3.9) is fulfilled with any  $D$ . So, applying (3.12) with  $D = d$  to the semigroup generated by the operator  $-\Delta_{\mathcal{M}} - \beta$ , we justify (7.2) for any  $\beta < 0$ . It extends to any values  $\beta < \beta_{\mathcal{M}}$  by the standard variational argument. One should only take into account that for all  $\beta < \beta_{\mathcal{M}}$  the quadratic forms

$$\int_{\mathcal{M}} (|\nabla u|^2 - \beta |u|^2) dx$$

generate mutually equivalent metrics on the Sobolev space  $H^1(\mathcal{M})$ .

The main issue in this type of problems is whether the estimate (7.2) remains valid for  $\beta = \beta_{\mathcal{M}}$ . Just such an estimate, rather than



(7.2) for  $\beta = 0$ , should be considered as the genuine generalization of the RLC inequality (2.1) to the operators on manifolds. The answer to this question is positive only in some special cases. The Hyperbolic Laplacian is one of these cases.

**7.2. Hyperbolic Laplacian.** Let us consider the  $d$ -dimensional Hyperbolic space  $\mathbb{H}^d$  for  $d \geq 3$ , in the upper half-space model. This means that  $\mathbb{H}^d$  is realized as  $\mathbb{R}_+^d := \mathbb{R}^{d-1} \times \mathbb{R}_+$ , with the metric

$$ds^2 = z^{-2}(|dy|^2 + dz^2), \quad y \in \mathbb{R}^{d-1}, \quad z \in \mathbb{R}_+.$$

The corresponding volume element is  $dv_{\text{hyp}} = z^{-d} dy dz$ . Recall that the Hyperbolic Laplacian is given by

$$\Delta_{\text{hyp}} = z^2(\Delta_y + \partial_z^2) - (d-2)z\partial_z,$$

where  $\Delta_y$  stands for the Euclidean Laplacian on  $\mathbb{R}^{d-1}$ . The bottom of  $-\Delta_{\text{hyp}}$  is the point  $\beta_0(d) = \frac{(d-1)^2}{4}$ .

The following result, which can be called the RLC estimate for the Hyperbolic Laplacian, was obtained in [16].

**Theorem 7.2.** *Let  $d \geq 3$  and  $0 \leq V \in L^{\frac{d}{2}}(\mathbb{H}^d)$ . Then*

$$N_-(\beta_0(d); -\Delta_{\text{hyp}} - \alpha V) \leq C(d)\alpha^{\frac{d}{2}} \int_{\mathbb{H}^d} V^{\frac{d}{2}} dv_{\text{hyp}}.$$

For the proof, one considers the quadratic form of  $-\Delta_{\text{hyp}}$  which is

$$Q[u] = \int_{\mathbb{H}^d} (|\nabla_{\text{hyp}} u|^2 - \beta_0(d)|u|^2) dv_{\text{hyp}} = \int_{\mathbb{R}_+^d} (|\nabla u|^2 - \beta_0(d)|u|^2) z^{2-d} dy dz.$$

The function  $\phi(y, z) = z^{\frac{d-1}{2}}$  satisfies the equation  $-\Delta_{\text{hyp}}\phi = \beta_0(d)\phi$ . The standard substitution  $u = w\phi$  reduces  $Q[u]$  to the form

$$Q[u] = \int_{\mathbb{R}_+^d} |\nabla w|^2 z dy dz.$$

For this quadratic form the lower bound is already  $\beta = 0$ . Now the global Sobolev inequality, which allows to apply Theorem 3.1 and leads to the estimate in Theorem 7.2, follows from [19], Corollary 2.1.6/3.

## 8. OPERATORS ON MANIFOLDS: BEYOND THEOREM 3.4.

A theory, allowing one to describe the potentials  $V$  on a general manifold, which ensure the semiclassical behavior  $N_-(\beta_{\mathcal{M}}; -\Delta_{\mathcal{M}} - \alpha V) = O(\alpha^{\frac{d}{2}})$ , does not exist. The situation simplifies if one has a more detailed information about the manifold, than that given by the values of the exponents  $\delta$  and  $D$  in the inequalities (3.8), (3.9). We illustrate

this by several examples. We start with the simple case of a compact manifold.

**Example 8.1.** Let  $\mathcal{M}$  be a compact and connected Riemannian manifold of dimension  $d \geq 3$ . Then the spectrum of  $\mathbf{A} = -\Delta_{\mathcal{M}}$  is discrete. The number  $l_0 = 0$  is a simple eigenvalue of  $-\Delta_{\mathcal{M}}$ , the corresponding eigenspace  $\mathcal{F}$  is formed by constant functions on  $\mathcal{M}$ . So, we have  $\beta_{-\Delta_{\mathcal{M}}} = 0$ . The estimate (7.2) for  $\beta = 0$  certainly fails, which immediately follows from the analytic perturbation theory: indeed, it shows that for any non-trivial  $V \geq 0$  and any  $\alpha > 0$  the operator  $-\Delta_{\mathcal{M}} - \alpha V$  has at least one negative eigenvalue. On the contrary, (7.2) with  $\beta = 0$  would give  $N_-(0; -\Delta_{\mathcal{M}} - \alpha V) = 0$  for  $\alpha$  sufficiently small.

It is easy to show that instead of (7.2) we have in this example:

$$N_-(0; -\Delta_{\mathcal{M}} - \alpha V) \leq 1 + C(\mathcal{M})\alpha^{\frac{d}{2}} \int_{\mathcal{M}} V^{\frac{d}{2}} dx. \quad (8.1)$$

The estimate (8.1) has the same properties as the RLC estimate for  $\mathbb{R}^d$ : it gives the correct order in  $\alpha \rightarrow \infty$  and it involves the sharp class of potentials for which this order is correct.

In general, for noncompact manifolds, one or both of these properties can be lost and some additional reasoning must be used.

In the case  $d > D > 2$  the estimate (3.10) implies (3.3) with  $2p = d$  for any compactly supported  $V$ , and, similarly to the case of a compact manifold, this result is sharp. Next, if the support of  $V$  has infinite measure and  $V \in L^{\frac{d}{2}} \cap L^{\frac{D}{2}}$ , neither of the terms in (3.10) majorizes the other one for a fixed  $\alpha$ , however when  $\alpha \rightarrow \infty$ , the first term in (3.10) dominates. This indicates that it is possible to relax the condition of finiteness of the expression in (3.10) and still have the semiclassical order in large coupling parameter. This difference in the dimensions  $d, D$  may generate an additional channel, which can contribute to the behavior of  $N_-(\beta_{\mathcal{M}}; -\Delta_{\mathcal{M}} - \alpha V)$  in a non-trivial way.

In the next example  $\mathcal{M}$  is a product manifold.

**Example 8.2.** Let  $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^m$ , where  $\mathcal{M}_0$  is a compact, connected smooth manifold of dimension  $d - m$ . We suppose that  $d \geq 3$ . Denote the points on  $\mathcal{M}$  as  $(x, y)$  where  $x \in \mathcal{M}_0$  and  $y \in \mathbb{R}^m$ ; further,  $dx$ ,  $dy$  stand for the volume element on  $\mathbb{R}^m$  and on  $\mathcal{M}_0$  respectively, then the volume element on  $\mathcal{M}$  is  $d\sigma = dx dy$ . The heat kernel on  $\mathcal{M}$  is the product of heat kernels on  $\mathcal{M}_0$  and  $\mathbb{R}^m$ , and easy calculations show that here  $\delta = d$ ,  $D = m$ . If  $m > 2$  the estimate (3.10) applies, as soon as  $V \in L^{\frac{d}{2}} \cap L^{\frac{m}{2}}$ , however this condition on  $V$  is not sharp since the first term in (3.10) majorizes the second one as  $\alpha \rightarrow \infty$ . For  $m \leq 2$

we simply cannot apply (3.10). The reasoning below demonstrates a typical way to handle such situations.

The Laplacian on  $\mathcal{M}_0$  has the lowest eigenvalue  $l_0 = 0$ , simple, with the corresponding eigenspace consisting of constants. Let  $l_1$  be the first nonzero eigenvalue on  $\mathcal{M}_0$ . Consider the orthogonal decomposition of the space  $L^2(\mathcal{M})$ ,

$$L^2(\mathcal{M}) = \mathfrak{F} \oplus \tilde{L}^2(\mathcal{M}), \quad (8.2)$$

where  $\mathfrak{F}$  consists of functions depending only on  $y$ , i.e.  $u(x, y) = v(y)$ ,  $v \in L^2(\mathbb{R}^m)$ . Given a function  $u \in L^2(\mathcal{M})$ , its orthogonal projection onto  $\mathfrak{F}$  is

$$v(y) = \frac{1}{\text{vol } \mathcal{M}_0} \int_{\mathcal{M}_0} u(x, y) dx,$$

which implies that  $\tilde{L}^2(\mathcal{M})$  consists of functions  $\tilde{u}(x, y)$  with zero integral over  $\mathcal{M}_0$  for almost all  $y \in \mathbb{R}^m$ . The decomposition (8.2) is orthogonal also in the metric of the Dirichlet integral,

$$\int_{\mathcal{M}} |\nabla u|^2 dx dy = \int_{\mathcal{M}} |\nabla \tilde{u}|^2 dx dy + \int_{\mathbb{R}^m} |\nabla v(y)|^2 dy.$$

Denote by  $\tilde{H}^1(\mathcal{M})$  the space of those  $\tilde{u} \in \tilde{L}^2(\mathcal{M})$  that belong to  $H^1(\mathcal{M})$ . On  $\tilde{H}^1(\mathcal{M})$  the metric, generated by the Dirichlet integral, is equivalent to the standard metric in  $H^1(\mathcal{M})$ :

$$\int_{\mathcal{M}} |\nabla \tilde{u}|^2 dx dy \geq \frac{1}{2} \left( \int_{\mathcal{M}} |\nabla \tilde{u}|^2 dx dy + l_1 \int_{\mathcal{M}} |\tilde{u}|^2 dx dy \right), \quad \forall \tilde{u} \in \tilde{H}^1(\mathcal{M}). \quad (8.3)$$

We also have

$$\int_{\mathcal{M}} V|u|^2 dx \leq 2 \left( \int_{\mathcal{M}} V|\tilde{u}|^2 dx + \int_{\mathbb{R}^m} W(y)|v(y)|^2 dy \right) \quad (8.4)$$

where the ‘effective potential’  $W(y)$  is given by

$$W(y) = \int_{\mathcal{M}_0} V(x, y) dy. \quad (8.5)$$

The inequalities (8.3), (8.4), being combined with the variational principle, show that

$$N_-(0; \Delta_{\mathcal{M}} - \alpha V) \leq N_-(-l_1; \Delta_{\mathcal{M}} - c\alpha V) + N_-(0; -\Delta_{\mathbb{R}^m} - c\alpha W), \quad (8.6)$$

where  $c > 0$  is some constant depending only on the value of  $l_1$ , and the second term corresponds to the Schrödinger operator on  $\mathbb{R}^m$ , with the potential  $-c\alpha W(y)$ . For the first term in (8.6) we can use the estimate (7.2). The appearing of the second term in (8.6) can be interpreted as opening of a new channel in the system under consideration. For estimating this term, we can use Theorem 4.1, 6.2, or 5.2, depending on

the dimension  $m$ . We would like to emphasize that here we need just the estimates of order  $O(\alpha^{\frac{d}{2}})$ . For the Laplacian on  $\mathbb{R}^m$  such estimates are non-semiclassical.

Moreover, suppose that  $V \in L^{\frac{d}{2}}$  but the effective potential  $W$  given by (8.5) satisfies the conditions of one of these theorems with some  $q > \frac{d}{2}$ . Then it may happen that the second term in (8.6) is stronger than the first one. In particular, if  $m \geq 3$  and the potential  $W$  is like in (4.5), this second term, in fact, gives the correct asymptotic behavior of the function  $N_-(0, \mathbf{H}_{\alpha V})$ .

Recall that for the operators on the half-line Theorem 5.2 gives the necessary and sufficient condition for the behavior  $N_-(0; \mathbf{H}_{\alpha V}) = O(\alpha^q)$  with a prescribed value of  $q > \frac{1}{2}$ ; this condition extends to the operators on the whole line in an obvious way. So, for  $m = 1$  the above construction gives more than for  $m \geq 2$ . Namely, it leads to the following result.

**Theorem 8.3.** *The two conditions:  $V \in L^{\frac{d}{2}}(\mathcal{M})$  and*

$$\#\{j \in \mathbb{Z} : 2^j \int_{2^j \leq |y| \leq 2^{j+1}} W(y) dy\} = O(l^{-\frac{d}{2}}), \quad l + l^{-1} \rightarrow \infty$$

*are necessary and sufficient for the semiclassical behavior of the function  $N_-(0; -\Delta_{\mathcal{M}} - \alpha V)$ , where  $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$  is a  $d$ -dimensional cylinder,  $d \geq 3$ .*

Note that the inclusion  $V \in L^{\frac{d}{2}}(\mathcal{M})$  does not imply any restrictions on the behavior of  $W$ . Actually, under some additional assumptions about  $W$  the function  $N_-(0; \mathbf{H}_{\alpha V})$  may have regular asymptotic behavior of order  $\alpha^{\frac{d}{2}}$  but with the asymptotic coefficient different from classical Weyl formula.

This kind of results can be easily extended to manifolds with cylindric ends.

In order to better understand the mechanism lying behind such two-term estimates, let us consider the free Laplacian  $-\Delta_{\mathcal{M}}$  in Example 8.2. Separation of variables shows that  $-\Delta_{\mathcal{M}}$  is unitary equivalent to the orthogonal sum of the operators  $-\Delta_{\mathbb{R}^m} + l_k$ ,  $k = 0, 1, \dots$  where  $l_k$  are the eigenvalues of  $-\Delta_{\mathcal{M}_0}$ ; recall that  $l_0 = 0$ . So, the structure of the spectrum of  $-\Delta_{\mathcal{M}}$  on  $[0, l_1)$  is determined by the  $m$ -dimensional Laplacian. This makes it clear, why the behavior of the function  $N_-(0; -\Delta_{\mathbb{R}^m} - \alpha W)$  in dimension  $m < d$  may affect the behavior of  $N_-(\beta_{\mathcal{M}}; -\Delta_{\mathcal{M}} - \alpha V)$  for the Laplacian on a manifold of dimension  $d$ .

The result of Theorem 7.1 shows that this effect does not appear for the function  $N_-(\beta; -\Delta_{\mathcal{M}} - \alpha V)$  with  $\beta < \beta_{\mathcal{M}}$ .

This can be considered as manifestation of the ‘threshold effect’ in this type of problems.

This effect exhibits in many other problems. One of them concerns the behavior of  $N_-(0; -\Delta - \alpha V)$  on  $\mathbb{R}^2$ , discussed in Section 6. Note that this is the problem where the effect of appearance of an additional channel was observed and explained for the first time, see [25] and [1]. It is worth noting also, that in the latter problem the mechanism behind this effect is rather latent. Indeed, unlike in Example 8.2, here removing the ‘bad’ subspace of radial functions does not lead to the shift of the spectrum of the unperturbed operator.

Another class of problems where the threshold effect has to be taken into account, concerns various periodic operators, perturbed by a decaying potential. In this connection, see the papers [2, 5].

One meets similar effects when studying the behavior of  $N_-(\beta; -\Delta_{\Omega} - \alpha V)$ , where  $\Delta_{\Omega}$  is the Dirichlet Laplacian in an unbounded domain  $\Omega \subset \mathbb{R}^d$ . For the corresponding heat kernel the estimate (3.8) with  $\delta = d$  always holds. Again, it may happen that the bottom of  $-\Delta_{\Omega}$  is a point  $\beta_0 > 0$ . Suppose  $d \geq 3$ , then for any  $\beta < \beta_0$  Theorem 7.1 applies. So, the problem consists in finding the estimates and the asymptotics of  $N_-(\beta_0; -\Delta_{\Omega} - \alpha V)$ . The general strategy here is the same as for manifolds, and examples like 8.2 can be easily constructed.

## 9. SCHRÖDINGER OPERATOR ON A LATTICE

The techniques based upon Theorems 3.1 and 3.2 applies also to the discrete Laplacian. Below we present some results for the simplest case, when the underlying measure space  $(\Omega, \sigma)$  is  $\mathbb{Z}^d$  with the standard counting measure, so that  $\sigma(E) = \#E$  for any subset  $E \subset \mathbb{Z}^d$ . For definiteness, we discuss only the case  $d \geq 3$ . The discrete Laplacian is

$$(\mathbf{A}_d u)(x) = \sum_j (u(x + 1_j) + u(x - 1_j) - 2u(x)), \quad x \in \mathbb{Z}^d,$$

where  $1_j$  is the multi-index with all zero entries except 1 in the position  $j$ . This is a bounded operator, and its spectrum is absolutely continuous and coincides with the segment  $[0, 2d]$ . The corresponding heat kernel can be found explicitly, it is bounded as  $t \rightarrow 0$  and is  $O(t^{-\frac{d}{2}})$  as  $t \rightarrow \infty$ , thus  $\delta = 0$  and  $D = d$ . The inequality (3.12) applies, and we obtain the discrete RLC estimate,

$$N_-(0; \mathbf{A}_d - \alpha V) \leq C \alpha^{\frac{d}{2}} \int_{\mathbb{Z}^d} V^{\frac{d}{2}} d\sigma, \quad \forall \alpha > 0; \quad d \geq 3.$$

On the contrary to the continuous case, this estimate can not be order-sharp, since the assumption  $V \in L^{\frac{d}{2}}(\mathbb{Z}^d)$  immediately yields

$$N_-(0; \mathbf{A}_d - \alpha V) = o(\alpha^{\frac{d}{2}}), \quad \alpha \rightarrow \infty.$$

Indeed, this is certainly true for any  $V$  with bounded support, since for such  $V$  the number  $N_-(0; \mathbf{A}_d - \alpha V)$  is no greater than the number  $\#\{x \in \mathbb{Z}^d : V(x) \neq 0\}$ . The set of all such  $V$  is dense in  $L^{\frac{d}{2}}(\mathbb{Z}^d)$ . Therefore, the result extends to all  $V \in L^{\frac{d}{2}}$ .

We do not know even a single example of a potential  $V$  on  $\mathbb{Z}^d$ , such that  $N_-(0; \mathbf{A}_d - \alpha V) = O(\alpha^{\frac{d}{2}})$  but  $\neq o(\alpha^{\frac{d}{2}})$ .

One more important difference with the continuous case is that for the discrete operators the behavior  $N_-(0; \mathbf{A}_d - \alpha V) = O(\alpha^q)$  with  $2q < d$  is possible; in the continuous case it never occurs in dimensions  $d \geq 3$  and, probably, also in  $d = 2$ . In  $d = 1$  the order  $O(\alpha^q)$  with  $2q < 1$  is possible, if one allows potentials which are distributions supported by a subset of zero Lebesgue measure.

For a given potential  $V \geq 0$  on  $\mathbb{Z}^d$ , one cannot formally use (3.11) with  $\delta = 0$ , since the value  $\delta = 0$  lies outside the set admissible by Theorem 3.4. However, by using the variational principle and (3.11) written for the potential  $V$  restricted to the set  $\{x : \alpha V(x) < 1\}$ , it is not difficult to show that in this particular case (3.11) holds for any  $\alpha > 0$  even with  $\delta = 0$ . If we introduce the distribution function of  $V$ ,

$$m(\tau) = \#\{x \in \mathbb{Z}^d : V(x) > \tau\}, \quad \tau > 0,$$

this line of reasoning leads to the inequality

$$N_-(0; \mathbf{A}_d - \alpha V) \leq C \left( m(2\alpha^{-1}) + \alpha^{\frac{d}{2}} \int_{\alpha V(x) < 1} V^{\frac{d}{2}} d\sigma \right) \quad (9.1)$$

By estimating the integral in (9.1), we come to the following result, which has no continuous analogue.

**Theorem 9.1.** *Suppose  $d \geq 3$ . Then for any  $\nu > 2$  the estimate holds*

$$N_-(0; \mathbf{A}_d - \alpha V) \leq C(d, \nu) \alpha^{\frac{d}{\nu}} \sup_{\tau > 0} \left( \tau^{\frac{d}{\nu}} \#\{x \in \mathbb{Z}^d : V(x) > \tau\} \right). \quad (9.2)$$

The class of discrete potentials  $V(x)$ , for which the functional on the right-hand side of (9.2) is finite, is nothing but the cone of all positive elements in the ‘weak’ space  $\ell_w^{\frac{d}{\nu}}(\mathbb{Z}^d)$ . The assumption  $\nu > 2$  yields

$$\ell_w^{\frac{d}{\nu}}(\mathbb{Z}^d) \subset L^{\frac{d}{2}}(\mathbb{Z}^d),$$

so that the estimate (9.1) applies. The inequality (9.2) gives a better estimate than (9.1), and it is possible to show that, unlike (9.1), it is order-sharp.

Theorem 9.1 applies to the potentials decaying no slower than  $|x|^{-\nu}$ ,  $\nu > 2$ , and gives the order  $O(\alpha^{\frac{d}{\nu}})$ . For potentials decaying more slowly (but still faster than  $|x|^{-2}$ ), so that the integral in (9.1) diverges, the following result applies. It is the direct analogue of Theorem 4.1; its proof is also based upon the interpolation theory.

**Theorem 9.2.** *Let  $d \geq 3$  and  $2q > d$ . Suppose the potential  $V(n) \geq 0$  is such that*

$$|V|_q^q := \sup_{\tau > 0} \left( \tau^q \int_{|x|^2 V(x) > \tau} \frac{d\sigma}{|x|^d} \right) < \infty.$$

*Then the estimate (4.2) holds for the operator  $\mathbf{H}_{\alpha V} = \mathbf{A}_d - \alpha V$ .*

In connection with Theorems 9.1 and 9.2 we note that for any  $\nu > 0$  the potential  $V(x) = (|x| + 1)^{-\nu}$  belongs to the class  $\ell_w^{\frac{d}{\nu}}(\mathbb{Z}^d)$ , and the potential  $V$  that for large  $|x|$  behaves as

$$V(x) = |x|^{-2} \log(|x|)^{-\frac{1}{q}}, \quad 2q > d,$$

meets the property  $|V|_q < \infty$ . So, these theorems embrace the cases of estimates of orders, respectively, smaller and larger than  $\alpha^{\frac{d}{2}}$ . It is unclear at the moment whether a sharp estimate of the order  $\alpha^{\frac{d}{2}}$  is possible. This indicates that the notion of 'semiclassical' order is not applicable here.

The above results can be extended to combinatorial Schrödinger operators on arbitrary infinite graphs, as soon as the heat kernel estimates (3.8), (3.9) are known with  $\delta = 0$ ,  $D > 2$ .

## 10. SOME UNSOLVED PROBLEMS

In this concluding section we list some problems in this field, which remain unsolved up to present. In our opinion, their solution would be important for the further progress in the field.

In the first place, this is the study of the Schrödinger operator on  $\mathbb{R}^2$ . Here we mean an exhaustive description of potentials  $V$  ensuring the semiclassical behavior  $N_-(0; \mathbf{H}_{\alpha V}) = O(\alpha)$ . As it was mentioned in Section 6, the situation here is unclear, and many natural conjectures fail to be true.

The next class of problems concerns manifolds. In particular, we believe that the class of  $d$ -dimensional manifolds,  $d \geq 3$ , for which the

structure of the potentials  $V$ , guaranteeing the semiclassical estimate  $N_-(\beta_{\mathcal{M}}; -\Delta_{\mathcal{M}} - \alpha V) = O(\alpha^{\frac{d}{2}})$ , can be exhaustively described, can be considerably widened compared with Theorem 8.3.

Theorem 9.1 indicates that the problems for the continuous and the discrete Schrödinger operators have rather different nature, and the expected results for these two parallel classes of operators should essentially differ. It would be useful to understand the discrete case up to a greater extent.

Finally, we mention the problems of the type discussed, for the metric graphs (quantum graphs, in other terminology), in particular for the metric trees. The few existing results, see, e.g., [8], still do not give the adequate understanding of the effects which appear when studying the Schrödinger operator on graphs.

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